Longevity Risk: Measurement and Application Perspectives

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Abstract

The paper presents a model involving an integrated analysis of demographic and financial risks for a portfolio of policies. In the case of life annuities, the impact of the longevity risk is studied, taking into account the interactions with the financial risk source; in particular the randomness in choosing projected mortality rates is considered in portfolio's and reserve's valuation.

Numerical examples illustrate the results, showing the behaviour of the projection risk.

Key words: Actuarial valuations, reserve, longevity risk, random projected mortality tables.

1 Introduction

In the general outlines of the life insurance business, it is well known that a great part of the issued policies are related to living benefits products, mostly characterised by single premium payments and long durations. In these cases companies' performances largely depend both on investment decisions and on demographic changes. In particular this last factor causes accidental errors and systematic deviations of the real data about deaths from their expected values. The systematic risk component of demographic nature is known as longevity risk: it is due to improvements in mortality

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trend and, in terms of survival function, is represented by the so called expansion phenomenon, that is the random advancement of the mode of the curve of deaths toward very old ages. Useful instruments for controlling the longevity risk are the projected mortality tables, constructed forecasting future mortality trends. It follows that the riskiness related to this kind of portfolios is many-sided in three components: the financial risk, due to changes in capital markets causing depreciation of the value of investments, the insurance risk, due to accidental deviations of the number of deaths from its anticipated values and the longevity risk. These aspects of riskiness involve consequences not only on the company's competitiveness through a correct choice of premium loading, but naturally on its solvency, too. As regards this last aspect, as well known, the system used in European Economic Area (EEA) is a simple solvency one that, acknowledging the necessity to set up a reserve for guaranteeing the solvency, settles the minimum solvency margin, consisting in a minimum margin requirement. As reported in [IAIS], the required minimum solvency margin can give only a static idea of the solvency situation for an insurance Company. On the other hand, it is opportune to point out that the risk management can understand the underlying risks and effects strategies to ensure that the overall risk profile is fronted. So, as explained in [IAIS], in order to asses an insurance company's situation about solvency, not only its minimum solvency margin, as stated in EEA, but also its risk profile constitute relevant information. From these considerations, a more adequate minimum solvency margin is desirable to adopt, in which the description and the analysis of technical and investment risks and in particular the measurement of the minimum required solvency margin these risks generate, are taken into account. It follows that an accurate study for identifying, analysing and measuring risks for insurance portfolios, and in particular for life annuity ones, is opportune in order to put into effects a correct policy of risk management. Risk management becomes in this way an essential component of the solvency assessment system, besides of the competing capacity of the company. The integrated analysis of demographic and financial risks and of their interactions in the global riskiness of a portfolio is, in this framework, fundamental in order to furnish a more detailed, useful and dynamic picture of the company, being able to survive as business in solvency conditions.

In this paper we want to give a contribute to the measurement of the longevity risk component; in particular we will present the projection risk, connected to the choice of the projected mortality table to use for portfolio valuations. For what precedes, the study of the impact of this risk on the portfolio's reserves seems particularly interesting, in a framework in which stochastic interest rates of return on investments and accidental deviations of mortality are taken into account.

In section 2, after a brief overview on longevity and projection risks, a measurement tool for the projection risk is presented.

In Section 3 the impact of the projection risk on reserves is evaluated and its behaviour for a life annuity portfolio is shown in some numerical illustrations. In section 4 some final considerations conclude the paper.

2 The uncertainty map

2.1 Longevity risk and projection risk

As regards living benefits, the actuary has to front two fundamental risk sources, that is the *investment risk* and the *demographic risk*.
The first one, due to the uncertainty in markets in which the company invests, consists in random fluctuations of the rates of return; as it is well known, the investment risk is a systematic risk component, the cash flows connected to the portfolio being discounted using the same rates.

The *demographic risk* is divided in two components: the *insurance risk* and the *longevity risk*. The insurance risk arises from accidental deviations of the number of deaths from its expected values, and it is a *pooling risk*, i.e. it can be mitigated by increasing the number of policies.

The longevity risk derives from improvements in mortality trend, which determine systematic deviations of the number of deaths from its expected values.

Really we observe the combined effect of more complex phenomena; in fact, at adult ages, one can note two relevant aspects for the survival functions: the higher concentration of deaths around the mode of the curve of deaths, and the random advancement of the mode of the curve of deaths towards the ultimate life time. The longevity risk comes out from this last aspect: its nature is systematic and the actuary fronts this risk component by means of *projected mortality tables*, that is tables obtained considering the improving mortality trend.

Moreover the choice of mortality rates with a certain projection level is uncertain, and it determines a model risk, called *projection risk*. Really a complete investigation of the effects due to the longevity risk can be reached only framing it in a stochastic point of view; in other words, the more concrete approach to this problem consists in studying the impact of the random improving dynamic of mortality rates and their effects on portfolio's management.

We have to take into account two levels of the problem: firstly the outcoming cash-flows depend on the insureds' life time, and the economic results are affected by decreasing mortality rates; on the other side we observe that the company's performance is strongly depending on the initial decisions about the investments, so, in perspective, reduced returns on financial investments could be obtained. In conclusion the combined effects due both to the randomness of financial incomes and the improvements in mortality trends could cause negative consequences in economic results and in solvency. So, for a correct managerial and institutional perspective, it is important to make an integrated analysis of the financial and demographic risk components; in this order of ideas in the following we will present a measure tool for the projection risk, considering a scenario involving also the stochastic returns obtained on the basis of an investment strategy.

2.2 A mesurement tool for the projection risk

Let T_i be the random variable representing the curtate-future-lifetime of the i-th life insured; let Z_i be the random variable representing the present value of a whole life annuity-immediate, of 1 unit payable at the end of each year while the i-th life aged (x) survives:

$$Z_i = \sum_{h=1,\ldots,T_i} \exp(-y(h)) \tag{1}$$

where

$$y(k) = \int_0^k \delta(s)ds,$$

$\delta(s)$ being the stochastic rate of return.

Let $Z(c)$ be the random variable representing the total present value of c identical annuities of the above described type:

$$Z(c) = \sum_{i=1}^{c} Z_i. \tag{2}$$

In order to quantify the impact of the randomness of the projection, taking into account also the presence of the investment risk and the insurance risk, we introduce the random survival function P, used for constructing the projected tables. The following result holds (cf. also [CDLS3])

Proposition 1 *Let us suppose that the random variables T_i are independent and identically distributed, while the random variables Z_i are identically distributed and independent, conditioning on the knowledge of $\{y(s)\}_{s=1,\ldots}$; moreover let the random variables T_i and $\{\delta(s)\}$ be mutually independent. Then $V[E[Z(c)|P]$ is a measure of the projection risk.*

Proof: Recalling known properties of conditional probability distributions, it holds

$$V[Z(c)] = V[E[Z(c)|P]] + E[V[Z(c)|P]]; \tag{3}$$

while the variance is an indicator of the global risk, indistincly including all risk sources, $V[E[Z(c)|P]]$ is a measure of the variability of $Z(c)$ due to the effect of the randomness of the projection, the effect of the other risk components (random rates of return and mortality random deviations) having been averaged out (cf. [CDLS3] and [DLOS]). In particular we get

$$V[E[Z(c)|P]] = V[E[\sum_{i=1}^{c} Z_i|P]] = V[cE[\sum_{h=1}^{T_i} e^{-y(h)}|P]] = \tag{4}$$

$$= c^2 V_P[\sum_{h=1}^{\omega-1-x} {}_h p_x E[e^{-y(h)}]].$$

Now, considering the average cost per policy of the portfolio, $\frac{Z(c)}{c}$, by means of straightforward calculations we obtain

$$V[E[\frac{Z(c)}{c}|P]] = V_P[\sum_{h=1}^{\omega-1-x} {}_h p_x E[e^{-y(h)}]],$$

so the above risk index is not dependent on the portfolio's size, coherently with the systematic nature of the projection risk. ∎

Remark. In the actuarial literature there exist risk indexes, variously used in valuation, pricing and solvency problems. Among them we recall in particular the index furnished by the coefficient of variation of the present value of the cash flows connected to the portfolio (cf. for example [OP]), given by

$$r = \frac{\sigma(Z)}{E(Z)}. \tag{5}$$

Such index gives useful information about risk management policy, as well as about reserves constituting and loading amounts establishing. In fact, as regards this last topic (cf. also [OP]), besides implicit loadings, to which the employment of opportune projected mortality tables contributes, also explicit loadings are assigned for fronting adverse deviations of Z with respect to its mean value. Recalling the results in [O] and [OP], letting the loading m given by

$$m = \mu\sigma(Z),$$

where μ is a fixed boundary which cannot be overcome by the random variable $\frac{Z-E[Z]}{\sigma[Z]}$, it is easy to deduce the behaviour of the rate of loading by the study of the parameter r defined by the following formula

$$\frac{m}{E[Z]} = \frac{\mu\sigma[Z]}{E[Z]} = \mu r.$$

In particular (cf. [CDLS3]), remembering the first moment of Z

$$E[Z(c)] = E[\sum_{i=1}^{c} Z_i] = \sum_{i=1}^{c} E[Z_i] = cE[Z_i] = c\sum_{h=1}^{\infty} {}_hp_x E[e^{-y(h)}],$$

we get (see Appendix A.1)

$$\left(\frac{\sigma[Z]}{E[Z]}\right)^2 = \tag{6}$$

$$= \frac{V\left[E\left[Z(c)|P\right]\right] + E\left[V\left[E\left[Z(c)|y(h)\right]|P\right]\right] + E\left[E\left[V\left[Z(c)|y(h)\right]|P\right]\right]}{(E[Z])^2}$$

hence

$$\left(\frac{\sigma[Z]}{E[Z]}\right)^2 = \frac{V_P[\sum_{h=1}^{\omega-1-x} {}_hp_x E[e^{-y(h)}]]}{\left(E[Z_i]\right)^2} +$$

$$+\frac{E_P\left[\sum_{h=1}^{\omega-1-x}\sum_{k=1}^{\omega-1-x} {}_hp_x\ {}_kp_x\ cov(e^{-y(h)}, e^{-y(k)})\right]}{\left(E[Z_i]\right)^2} +$$

$$+\frac{E_P\left[E[V[Z_i \mid y(h)]]\right]}{c\left(E[Z_i]\right)^2}$$

We can observe that the first term on the right hand side in (6) provides an estimate of the explicit loading connected to the systematic risk arising from the projection randomness and, according to our expectation, it does not depend on c. The second term on the right hand side in (6) gives an estimate of the explicit loading connected to the systematic risk caused by the randomness of the rates of return and it is independent on c; finally the last term on the right hand side in (6) furnishes an estimate of the explicit loading connected to the risk due to the mortality randomness and, consequently, it is inversely proportional with respect to the number of policies c.

3 Risk valuations concerning the reserves

3.1 The impact of the projection risk

As it is well known, the *net premium reserve* at time t is defined as the conditional expectation of the difference at time t between the present value of future benefits and the present value of future premiums, given the present state of the policy (i.e., denoting by T the random future lifetime of an insured aged x, given that $T > t$). The above difference is represented by the random variable ${}_tL$, named *prospective loss*.

As explained in the introduction, studying the impact of the various risk factors on the reserves allows actuaries to get significant information about the company's performance.

Let us consider a whole life annuity-immediate, of 1 unit payable at the end of each year while the life aged (x) survives; the prospective loss at time t has the following expression

$$ {}_tL = \sum_{i=1}^{T^{(x+t)}} e^{-y(i)} = \sum_{i=1}^{T^{(x+t)}} e^{-\int_0^i \delta_s ds}, $$

with $T^{(x+t)}$ the random variable representing the future lifetime of an insured aged $x+t$, and δ_s being the force of interest as defined in section 2.2.

Let us consider a portfolio of c identical life annuity of the above mentioned type and let us indicate by $T_i^{(x+t)}$ the random variable representing the curtate future lifetime of the i-th life insured aged $x+t$ $(t = 0, 1, \ldots)$. In the following we will denote by ${}_tL^{(i)}$ the prospective loss at time t for the i-th life anuuity and by ${}_t\mathbf{L}$ the prospective loss for the entire portfolio. In particular, denoted by $N^x(t)$ the number of survivors at time t in the ambit of the c insured aged x at issue, it holds

$$ {}_t\mathbf{L} = N^x(t) \sum_{r=1}^{\infty} {}_rp_{x+t} e^{-y(r)}. \tag{7} $$

Now we want to quantify the impact of the randomness of the projection on the reserve, taking into account also the presence of the investment risk and the insurance risk. Recalling the notations introduced in Section 2.2, the following result holds

Proposition 2 *Under the hypotheses of Proposition 1, $V[E[{}_t\mathbf{L}|P]$ is a measure of the projection risk.*

Proof: Since

$$ V[{}_t\mathbf{L}] = V[E[{}_t\mathbf{L}|P]] + E[V[{}_t\mathbf{L}|P]] \tag{8} $$

reasoning as for the Proposition 1, we observe that $V[E[_t\mathbf{L}|P]]$ is a measure of the variability of $_t\mathbf{L}$ due to the effect of the randomness of the projection, the effect of the other risk components (random rates of return and mortality random deviations) having been averaged out. We have

$$V[E[_t\mathbf{L}|P]] = V\left[E\left[N^x(t)\sum_{r=1}^{\infty} {_rp_{x+t}}\ e^{-y(r)}|P\right]\right] = \tag{9}$$

$$= V_P\left[c\ {_tp_x}\sum_{r=1}^{\infty} {_rp_{x+t}}E[e^{-y(r)}]\right].$$

■

Remark. Formula (8) allows us to quantify the reserve's volatility due to both the oscillations of the rate of return and the random mortality's deviations, taking into account also the interactions with the projection risk. In fact the second term on the right hand side in (8) can be again split as follows

$$E[V[_t\mathbf{L}|P]] = E[V[E[_t\mathbf{L}|\{y(h)\}]|P]] + E[E[V[_t\mathbf{L}|\{y(h)\}]|P]]; \tag{10}$$

in particular we recognize that the first term on the right hand side is a measure of the investment risk, while the second one is a measure of the insurance risk. We calculate their expressions as follows

$$E[V[E[_t\mathbf{L}|\{y(h)\}]|P]] = E\left[V\left[E\left[N^x(t)\sum_{r=1}^{\infty} {_rp_{x+t}}\ e^{-y(r)}|\{y(h)\}\right]|P\right]\right] =$$

$$= (c\ {_tp_x})^2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} E[{_rp_{x+t}}\ {_sp_{x+t}}]cov(e^{-y(h)}, e^{-y(k)})$$

and

$$E[E[V[_t\mathbf{L}|\{y(h)\}]|P]] = E\left[E\left[V\left[N^x(t)\sum_{r=1}^{\infty} {_rp_{x+t}}\ e^{-y(r)}|\{y(h)\}\right]|P\right]\right] =$$

$$= cE\left[{_tp_x}(1 -_t p_x)\sum_{r=1}^{\infty} {_rp_{x+t}}\ E[e^{-y(r)}]\right].$$

3.2 Numerical applications

In order to give a numerical example involving the projection risk, we consider a portfolio of identical whole life annuity-immediate; we assume a stochastic scenario for the behaviour of the instantaneous rate of return, which is described as sum of two components: a deterministic one, estimated on the basis of the investment strategy, and a stochastic one, described by an Ornstein-Uhlenbeck process, with positive parameters β and σ and initial position equal to zero, apt to describe the deviations of the global rate of return from its anticipated values (see Appendix A.2). In our calculation we consider a constant deterministic component equal to 0.08, and for the stochastic process we consider a drift parameter $\beta = 0.12$ and a diffusion coefficient $\sigma = 0.006$.

We assume that the future lifetime of an insured aged $x = 0$ is represented by means of the Weibull model, with survival function

$$s(x) = e^{-(\frac{x}{\alpha})^{\gamma}}$$

α and γ being positive constant parameters. Choosing opportune values of the parameters α and γ it is possible to obtain projected tables with increasing survival probabilities; in particular, remembering [O], we construct a pessimistic, a realistic and an optimistic table, on the basis of the couples of parameters (83.5;8), (85.32;9.15), (87.0;10.45). Moreover we suppose that the pessimistic, the realistic and the optimistic projections have respectively probability 0.2, 0.6, 0.2 to be chosen.

In figures 1 and 2 we can observe the behaviour of the projection risk for the reserve calculated, respectively, at time $t = 10$ and $t = 30$, as function of the age at issue and the policy number. For every fixed value of the age at issue x, it increases with c; in the case of a reserve calculated in $t = 10$, the projection risk, for every fixed value of c, increases with x until the value x is a bit less than 60, after which it decreases; in the case of a reserve calculated in $t = 30$, the projection risk, for every fixed value of c, increases with x until the value x is a bit less than 40, after which it decreases.

Moreover in figure 3 we can see that, for a portfolio of $c = 1000$ policies, the younger the insured is (at issue), the more the projection risk reaches the hit for very high values of t. Figures 4 and 5 explain two particular cases of the projection risk when $x = 20$ and $x = 40$. The behaviour of the projection risk for $x \geq 70$ is due to the interactions between x and the time duration t considered in the projected survival probability ${}_tp_x$, and the possible further annuity duration; on the contrary for young ages there are low differences between the survival probabilities of the projected tables we consider; such differences, for young ages, produce an initial very small risk projection, increasing with age.

An analogous behaviour occurs when the roles of x and t are interchanged.

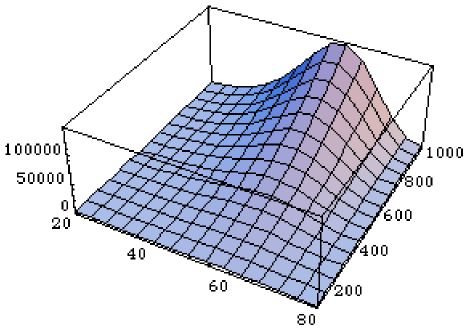

figure 1:
Projection risk for ${}_{10}\mathbf{L}$
$x = 20, ..., 80$; $c = 100, ..., 1000$

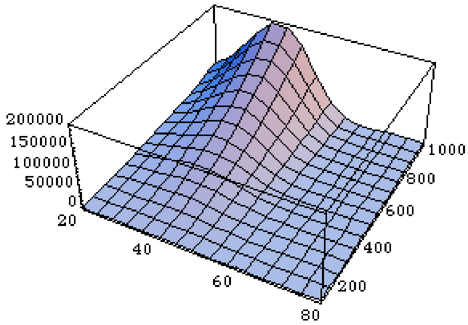

figure 2:
Projection risk for ${}_{30}\mathbf{L}$
$x = 20, ..., 80$; $c = 100, ..., 1000$

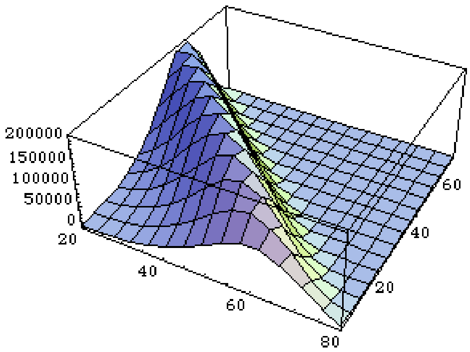

figure 3
Projection risk for ${}_{t}\mathbf{L}$
$x = 20, ..., 80$; $t = 10, ..., 70$; $c = 1000$

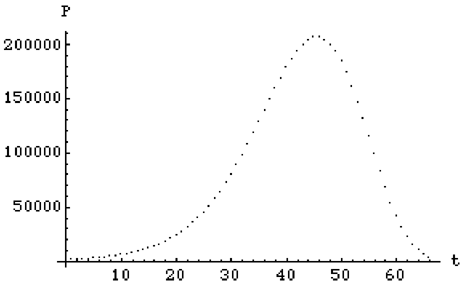

figure 4
Projection risk $x = 20, t = 5, ..., 60$

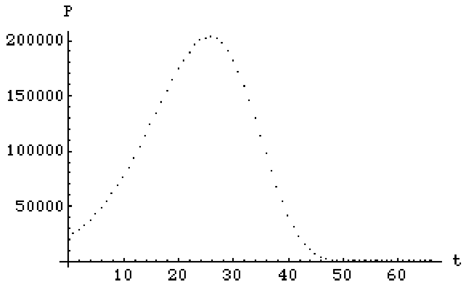

figure 5
Projection risk $x = 40, t = 5, ..., 60$

4 Conclusions

The importance of a careful consideration of the human life progressive extension when considering portfolios' risk valuation is by now well known. In fact this phenomenon is relevant in order to correctly lead the company's management both in terms of solvency and of premium loading estimation. Throughout the paper we focused on the projection risk, that is the model risk deriving from the choice of the projected mortality table to use for portfolio valuations. The projection risk is considered in a framework in which both interest and mortality are random. In particular, for the stochastic evolution of the interest rates of return on investments made by the company, an O.U. process is used, referred to the deviations of the real rates from the deterministic values anticipated for them. In the paper the impact of the projection risk on portfolios' reserve is analysed. The reserve volatility quantification is presented in its three components; in particular the projection risk for the reserve is graphically represented both as function of the age at issue and of the number of policies in portfolio, and as function of the age at issue and of the time of reserve valuation.

The practical implications of the above study come true in the possibility of an integrated analysis of demographic and financial risks and of their interactions in the global risk of a portfolio. Referring to the case of the reserves, the above analysis provide results useful for a more detailed and dynamic picture of the company's performances. In the ambit of a solvency analysis, a further research line could suggest operative links between the risk parameters presented and the minimum solvency margin required by the EEA's directives.

References

[BF1] BEEKMAN J.A., FUELLING C.P.: *Interest and Mortality Randomness in Some Annuities*; Insurance Math. Econom. (1990), 185-196.

[BF2] BEEKMAN J.A., FUELLING C.P.: *Extra Randomness in Certain Annuity Models*. Insurance Math. Econom. **10**, (1991), 275-287.

[BGHJN] BOWERS N.L., GERBER H.U., HICKMAN J.C., JONES D.A., NESBITT C.J.: "Actuarial Mathematics"; Society of Actuaries, Schaumburg (1986).

[CDLS1] COPPOLA M., DI LORENZO E., SIBILLO M.: *Risk Sources in a life Annuities Portfolio: Decomposition and Measurement Tools*; Journal of Actuarial Practice, vol. 8 (2000).

[CDLS2] COPPOLA M., DI LORENZO E., SIBILLO M.: *Investment Risk and Longevity Risk in a Life Annuities Portfolio*; Proceedings of the "II Italian-Spanish Conference on Financial Mathematics", Napoli (1999).

[CDLS3] COPPOLA M., DI LORENZO E., SIBILLO M.: *Further Remarks on Risk Sources Measuring in the Case of a Life Annuity Portfolio*; to appear on Journal of Actuarial Practice Vol. 10 (2003).

[DLOS] DI LORENZO E., ORLANDO A., SIBILLO M.: *Financial and Demographic Randomness Measuring: the Case of a Life Annuity Portfolio*; Proceedings of "ADSMDA 2001 10^{th} International Symposium on Applied Stochastic Models and Data Analysis", Govaert, Janssen and Limnios Editors, Compiegne (France), (2001).

[DLST] DI LORENZO E., SIBILLO M., TESSITORE G.: *A Stochastic Model for Financial Evaluations. Applications to Actuarial Contracts*; Appl. Stochastic Models Bus. Ind., **15**, (1999).

[F1] FREES E.W.: *Relative Importance of Risk Sources in Insurance Systems*; NAAJ (1998), 34-52.

[IAIS] International Association of Insurance Supervisor.: *On solvency, solvency assessments and actuarial issues*; An IAIS Issues Paper (final version) (2000).

[KB] KEFITZ N., BEEKMAN J.A.: *Demography through Problems*; Springer-Verlag, New York (1984).

[MP] MAROCCO P., PITACCO E.: *Longevity Risk and Life Annuity Reinsurance*; Transaction of the 26th International Congress of Actuaries, Birmingham, vol. **6** (1998), 453-479.

[O] OLIVIERI A.: *Per una quantificazione del longevity risk*; Atti Convegno AMASES, Genova (1998), 383-399.

[OP] OLIVIERI A., PITACCO E.: *Funding Sickness Benefits for the Elderly*: Dipartimento di Matematica Applicata alle Scienze Ec., Stat. e Att. "B. De Finetti", Trieste (1999).

[PW] PANDIT S.M, WU S.M.: "Time Series and System Analysis with Applications"; John Wiley and Sons (1983).

[P] PARKER G.: *Stochastic Analysis of the Interaction Between Investment and Insurance Risks*; NAAJ (1997), 55-84.

[U] UP-94 TASK FORCE: *The 1994 Uninsured Pensioner Mortality Table*; Transactions of the Society of Actuaries, (1996), Volume XLVII, 89-132.

Appendix

A.1 Formula (6) gives tools for measuring the contribution of the projection risk to the global risk, together with those ones referred to the investment risk, due to the fluctuations of the rates of return, and the insurance risk, due to the accidental deviations of the number of deaths from the expected values. Formula (6) can be justified recalling some results proved in [CDLS3]. In fact in [CDLS3], denoting by P the random survival function used for constructing the projected survival probabilities, the variance of the average cost per policy $\frac{Z(c)}{c}$ is decomposed in two components

$$V[\frac{Z(c)}{c}] = V[E[\frac{Z(c)}{c}|P]] + E[V[\frac{Z(c)}{c}|P]]). \qquad (A1)$$

The first term on the right hand side in (A1) is a measure of the randomness of the projection, the effect of the other risk components (random rates of return and mortality random deviations) having been averaged out. The second term is an average, taken over P, of the variability of $\frac{Z(c)}{c}$ due to both the rates of return and the mortality deviations.

Now considering $E[V[\frac{Z(c)}{c}|P]]$, we observe that the variable $V[\frac{Z(c)}{c}|P]$ is affected by two risk causes, i.e. the randomness of the rate of return and the random lifetime duration (that is the accidental deviations of the number of deaths); so we again split the expression of $V[\frac{Z(c)}{c}|P]$ with respect to the stochastic rates of return $\{y(h)\}_{h=1}^{\infty}$

$$E[V[\frac{Z(c)}{c}|P]] = E[V[E[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]] + $$

$$+ E[E[V[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]. \qquad (A2)$$

Reasoning as for (A1), it is possible to deduce that $E[V[E[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$ is a measure of the investment risk, and $E[E[V[\frac{Z(c)}{c}|\{y(h)\}_{h=1}^{\infty}]|P]]$ is a measure of the insurance risk.

A.2 Referring to the evolution in time of the instantaneous rate of return $\delta(t)$, in section 3.2 we have considered the sum of two components: a deterministic one, $r(t)$, representing the baseline obtained on the basis of the experience related to the results of the investment policy; a stochastic one, $X(t)$, which describes the deviations of the global rate of return from its anticipated values.

Since we fix an investment strategy for the considered time horizon, we do not need to consider a term structure model.

$X(t)$ is an Ornstein-Uhlenbeck process, with parameters $\beta > 0$ and $\sigma > 0$, and initial position $X(0) = 0$, solution of the following stochastic differential equation:

$$dX(t) = -\beta X(t)dt + \sigma dW(t), \qquad (A3)$$

where $W(t)$ is a standard Wiener process.

$r(t)$ is obtained using historical and economic information, whilst $X(t)$ expresses the deviations between the real rate and the anticipated one.

The stochastic present value at time 0 of a payment of one monetary unit at time t is:

$$e^{-y(t)} = e^{-\int_0^t \delta(s)ds} e^{-\int_0^t (r(s)+X(s))ds} = v(t)F(t),$$

where $v(t)$ is the deterministic discounting factor and $F(t)$ is the stochastic discounting factor (cf. [DLST]) lognormally distributed with parameters $-E[\int_0^t X(s)ds]$ and $var[\int_0^t X(s)ds]$.